(2)

Q.2 2015 Starting from B-E distribution function deduce the Plank radiation formula.

As photons can either be emitted or absorbed by the atoms on interacting with matter, therefore the number of photons is not constant. Hence the condition $\sum_i dn_i = 0$ is not valid.

For this reason the value of α in B-E distribution law is zero. Hence –

B-E distⁿ law, $n_i = \frac{g_i}{(e^{\alpha + E_i/KT}) - 1}$ becomes-

$$n_i = \frac{g_i}{e^{E_i/KT} - 1} \quad \longrightarrow (1)$$

Replacing $g_i = g(E)dE$ and $n_i = dn$ we have –

$$dn = \frac{g(E)\,dE}{e^{E/KT} - 1} \quad \longrightarrow (2)$$

Again $E = h\nu$ so the value of $g(E)\,dE$ can be equal to $g(\nu)\,d\nu$. The term $g(\nu)\,d\nu$ is the number of oscillatory modes in the frequency range $d\nu$ with energy dE.

The number of states in black body radiation in the frequency range ν and $\nu + d\nu$ can be obtained by calculating the spherical vol. of the sphere of radii →

$\frac{h(\nu+d\nu)}{c}$ and $\frac{h\nu}{c}$, i.e vol. betⁿ this two radii.

∴ The volume of the spherical cell –

$$g(\nu)\,d\nu = \frac{4}{3}\pi \frac{h^3}{c^3}(\nu+d\nu)^3 - \frac{4}{3}\pi \frac{h^3}{c^3}\nu^3$$

$$= \frac{4}{3}\pi \frac{h^3}{c^3}\left[\nu^3 + 3\nu^2 d\nu + \cdots - \nu^3\right]$$

$$= \frac{4}{3}\pi \frac{h^3}{c^3}\cdot 3\nu^2 d\nu$$

$$= 4\pi \frac{h^3}{c^3}\nu^2 d\nu \longrightarrow (3)$$

Again phase space has a vol. $V = h^3$.
Now doubling the eqn (2) due to the polarization of photon or two mode of propagation for each photon we have the radiation betⁿ the range ν and $\nu+d\nu$ as –

$$g(\nu)\,d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu \longrightarrow (4)$$

From eqn (2) and (4) we get –

$$dn = \left(\frac{8\pi V}{c^3}\nu^2 d\nu\right)\frac{1}{e^{h\nu/kT}-1} \rightarrow (5)$$

For dn photons in frequency range ν, and $\nu+d\nu$, the energy is equal to $(h\nu)dn$.

∴ The energy per unit vol. $= \frac{(h\nu)\,dn}{V}$

So, the energy density distribution for black body radiation is given by –

$$E(\nu) = \frac{(h\nu)\,dn}{V\,d\nu} \rightarrow (6)$$

putting the value of eqn (5) in eqn (6) we get -

$$E(\nu) = \frac{8\pi V \nu^2 d\nu \cdot h\nu}{c^3 V d\nu \left(e^{h\nu/kT} - 1\right)}$$

$$= \left(\frac{8\pi h \nu^3}{c^3}\right)\left(\frac{1}{e^{h\nu/kT} - 1}\right) \rightarrow (7)$$

The eqn (7) is the Planks radiation law for black body radiation.

Eqn (7) can be written in terms of frequency as -

$$E(\nu) d\nu = \frac{8\pi h\nu^3}{c^3}\left[\frac{1}{e^{h\nu/kT} - 1}\right] d\nu$$

Q.3 What is Fermi energy? Derive an
2015 expression for Fermi energy for an
electron gas.

In the case of Fermi-Dirac statistics, the accumulation of particles at the ground level is not allowed and at temperature $T=0$, the particles occupy the lowest energy levels upto E_F. Here, the energy E_F gives the indication of the maximum energy of the fermions of the system. The energy E_F is called Fermi energy.

According to F-D distribution law -

$$n_i = \frac{g_i}{e^{(E_i - E_F)/kT} + 1} \rightarrow (1)$$

Since the energy of the electron in the conduction band is continuous, the term g_i is replaced by $g(E)\,dE$, where dn electrons have energy in the range E and $E+dE$.

So, eqn (1) becomes -

$$dn = \frac{g(E)\,dE}{e^{(E_i - E_F)/kT} + 1} \rightarrow 2)$$

As an electron has spin $\pm\frac{1}{2}$, the total number of states in the sphere is twice. The term $\frac{V}{(2\pi)^3}$ refers to the translational states per unit volume in the Fermi space. So, Fermi sphere of radius k_F has the total

number of particles accomodated i.e.
The number of allowed energy values
betn E and E+dE. is -

$$n = \frac{2\frac{V}{(2\pi)^3}\left(\frac{4}{3}\pi(k_F)^3\right)}{e^{(E_i-E_F)/kT}+1} \rightarrow (3)$$

Again, $E = \frac{p^2}{2m} = \frac{\left(\frac{h}{2\pi}\right)^2 (k_F)^2}{2m}$

$$\therefore k_F = \frac{p}{h/2\pi} = \frac{2\pi p}{h} \rightarrow (4)$$

Hence eqn (4) became

$$n = \frac{2\frac{V}{(2\pi)^3}\left(\frac{4}{3}\pi\left(\frac{2\pi p}{h}\right)^2\right)}{e^{(E_i-E_K)/kT}+1}$$

$$= \frac{2\frac{V}{(2\pi)^3}\left[\frac{4}{3}\pi\left(\frac{2\pi}{h}\right)^3(2mE)^{3/2}\right]}{e^{(E_i-E_K)/kT}+1} \quad \Bigg| \because p=(2mE)^{1/2}$$

$$= \frac{\left(\frac{8\pi V}{3h^3}\right)(2m)(2m)^{1/2}E^{3/2}}{e^{(E_i-E_K)/kT}+1} \rightarrow (5)$$

Differentiating eqn (5) we get -

$$dn = \frac{\left(\frac{8\pi V}{3h^3}\right)(2m)(2m)^{1/2}\cdot\frac{3}{2}E^{1/2}\,dE}{e^{(E_i-E_K)/kT}+1}$$

$$\Rightarrow \frac{dn}{dE} = \frac{\left(\frac{8\pi V(2m^3)^{1/2}}{h^3}\right)E^{1/2}}{e^{(E_i-E_K)/kT}+1} \rightarrow (6)$$

The eqn (6) represents the energy distn for free electrons. This is also called F-D formula of free fermions.

Q. Ex. Write short note on Fermi energy.

According to Heisenberg Uncertainty principle we have -

$$\Delta x \Delta p \sim h \longrightarrow (1)$$

The eqn (1) can be generalised to all three components as -

$$(\Delta x \Delta p_x)(\Delta y \Delta p_y)(\Delta z \Delta p_z) \sim h^3 \longrightarrow (2)$$

$$\Rightarrow (\Delta p_x \Delta p_y \Delta p_z) \sim \frac{h^3}{\Delta x \Delta y \Delta z} \longrightarrow (3)$$

Considering the momentum volume elements $(\Delta p_x \Delta p_y \Delta p_z)$ as a sphere of radius Δp.

$$\therefore (\Delta p_x \Delta p_y \Delta p_z) = \frac{4}{3}\pi(\Delta p)^3$$

$$\sim \frac{h^3}{\Delta V} \longrightarrow (4)$$

By virtue of the uncertainty principle, each electron occupies at least a volume ΔV and this electron can exist in either of the two possible spin orientations.

$\therefore$ if N is the number of electrons in a unit volume, $\Delta V = \frac{2}{N}$

Hence, $\frac{4}{3}\pi(\Delta p)^3 \sim \frac{h^3 N}{2}$

$$\therefore \Delta p \sim h\left(\frac{3N}{8\pi}\right)^{1/3} \longrightarrow (5)$$

Since ΔV is the minimum volume needed to house an electron, Δp is maximum as a consequence of the uncertainty principle.

So, we have $$\epsilon_F = \frac{(\Delta p)^2}{2m}$$

$$= \frac{h^2}{2m}\left(\frac{3N}{8\pi}\right)^{2/3} \longrightarrow (6)$$

The eqn (6) represent the Fermi energy if $V = 1$.