1.3 PLANCK'S QUANTUM THEORY

In order to explain the distribution of energy in the spectrum of a black body, Max Planck in 1900, put forward the quantum theory of radiation. He assumed that the atoms in the walls of a black body behave like simple harmonic oscillators, and each has a characteristic frequency of oscillation. In his theory he made the following two radical assumptions about the atomic oscillators:

(1) A simple harmonic oscillator cannot have any arbitrary values of energy but only those values of the total energy E that are given by the relation:

$$E = nh\nu \qquad ...(1)$$

where $n = 0, 1, 2, 3, ...$; n is called the quantum number, ν is the frequency of oscillation, and h is a universal constant called Planck's constant ($h = 6.626 \times 10^{-34}$ Js). In this relation $h\nu$ is the basic unit of energy and it is called a *quantum of energy*. Thus the relation shows that the total energy of an oscillator is *quantized*.

(2) As long as the oscillator has energy equal to one of the allowed values given by the relation $E = nh\nu$, it cannot emit or absorb energy. Therefore, the oscillator is said to be in a stationary state or a quantum state of energy. *The emission or absorption of energy occurs only when the oscillator jumps from one energy state to another.* If the oscillator jumps down from a higher energy state of quantum number n_2 to a lower energy state of quantum number n_1, the energy emitted is given by :

$$E_2 - E_1 = (n_2 - n_1)\, h\nu$$

if $n_2 - n_1 =$ one unit, then

$$E_2 - E_1 = h\nu$$

Similarly, an oscillator absorbs a quantum $h\nu$ of energy when it jumps up to its next higher energy state.

According to Planck the quantum theory is applicable only to the process of emission and absorption of radiant energy.

In 1905 Einstein extended Planck's quantum theory by assuming that a monochromatic radiation of frequency ν consists of a stream of photons each of energy $h\nu$ and the photons travel through space with the speed of light.

Planck's Radiation Law

On the basis of the quantum theory, Planck obtained the formula for an average energy of an oscillator:

$$E = \frac{h\nu}{e^{h\nu/kT} - 1} \qquad ... (2)$$

It can be shown that the number of oscillations or degrees of freedom per unit volume in the frequency range ν and $\nu + d\nu$ is given by :

$$N(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\, d\nu \qquad ... (3)$$

where c is the speed of light in vacuum.

Then assuming that the average value of the energies of the various modes of oscillation in black body radiation is given by Eq. (2), Planck obtained the equation.

$$U_\nu d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}\, d\nu \qquad ...(4)$$

where $U_\nu d\nu$ is the energy per unit volume in the frequency range ν and $\nu + d\nu$ and U_ν is the energy per unit volume per unit frequency range at frequency ν. In terms of the wavelength of the radiation this equation is:

$$U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda kT} - 1}\, d\lambda \qquad ...(5)$$

Eqs. (4) and (5) are two forms of Planck's radiation law.

When the values of U_λ as obtained from Eq. (5) for different values of λ are plotted against the corresponding values of λ we get curves as shown in Fig. 1.3. These curves agree very well with the experimental results over the whole range of wavelengths.

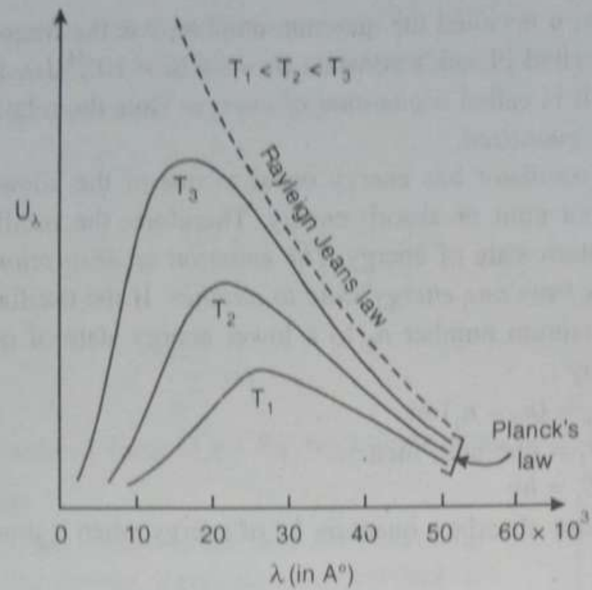

Fig. 1.3

Consequences of Planck's law

From Planck's law in the form of Eq. (5) the Rayleigh -Jeans law, Wien's law and the Stefan-Boltzmann formula are obtained as mathematical consequences:

(1) **Rayleigh-Jeans law**: For small values of $hc/\lambda kT$, *i.e.* in the region of long wavelengths, the exponential term in Eq. (5) can be expanded and retaining only the first term, we get

$$U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{hc/\lambda kT} d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda \qquad \text{... (6)}$$

This is the Rayleigh-Jeans law.

(2) **Wien's Radiation Formula:** In the region of low wavelengths $hc/\lambda kT$ becomes large. Hence 1 in the denominator on the right hand side of Eq. (5) can be neglected in comparison with the exponential term. Therefore, we get

$$U_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda kT}} d\lambda$$

$$= \frac{8\pi hc}{\lambda^5} e^{-hc/\lambda kT} d\lambda$$

It can be shown that for any black body, E_λ is related to U_λ at the same temperature by the equation :

$$E_\lambda = \frac{cU_\lambda}{4}, \quad \text{or} \quad U_\lambda = \frac{4E_\lambda}{c}$$

Now substituting for U_λ in the above equation, we get :

$$E_\lambda d\lambda = \frac{2\pi hc^2}{\lambda^5} e^{-hc/\lambda kT} d\lambda$$

$$= \frac{C_1}{\lambda^5} e^{-C_2/\lambda T} d\lambda \qquad \text{...(7)}$$

where $C_1 = 2\pi hc^2$ and $C_2 = hc/k$

Eq. (7) is Wien's radiation formula

(3) **Wien's displacement law :** From Planck's radiation law, we have :

$$U_\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda kT} - 1} \qquad \text{...(8)}$$

At constant temperature T of a black body, the wavelength λ_m at which the energy density is maximum is given by :

$$\left[\frac{dU_\lambda}{d\lambda}\right]_{\lambda_m} = 0$$

Taking logarithm of both the sides of Eq. (8), we have :

$$\log_e U_\lambda = \log_e(8\pi hc) - 5\log_e \lambda - \log_e\left(e^{hc/\lambda kT} - 1\right)$$

Differentiating this equation with respect to λ

$$\frac{1}{U_\lambda} \cdot \frac{dU_\lambda}{d\lambda} = 0 - \frac{5}{\lambda} - \left[\frac{1}{e^{hc/\lambda kT} - 1}\right]\left(e^{hc/\lambda kT}\right)\left(-\frac{hc}{\lambda^2 kT}\right)$$

$$= \frac{1}{\lambda}\left[-5 + \frac{hc}{\lambda kT}\left\{\frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT} - 1}\right\}\right]$$

At $\qquad \lambda = \lambda_m, \qquad \frac{dU_\lambda}{d\lambda} = 0$

$\therefore$

$$-5 + \frac{hc}{\lambda_m kT}\left\{\frac{e^{hc/\lambda_m kT}}{e^{hc/\lambda_m kT} - 1}\right\} = 0$$

Let $\qquad \frac{hc}{\lambda_m kT} = x,$

then $\qquad -5 + \frac{xe^x}{e^x - 1} = 0$

or $\qquad \frac{xe^x}{e^x - 1} = 5$

On solving this equation by trial and error method, we will get

$$x = 4.9651$$

i.e., $\qquad \frac{hc}{\lambda_m kT} = 4.9651$

$\therefore$

$$\lambda_m T = \frac{hc}{4.9651 \times k}$$

$$= \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{4.9651 \times 1.38 \times 10^{-23}} = \frac{6.62 \times 3 \times 10^{-3}}{4.9651 \times 1.38}$$

$$= \mathbf{2.898 \times 10^{-3}\ m\ K}$$

This relation is Wien's displacement law. The law can be used to determine the temperature of a black body by determining the wavelength λ_m at which the intensity of the radiation is maximum.

(4) Stefan-Boltzmann Law. On the basis of the experimental data of Tyndol and of Dulong and Pelit, J. Stefan in 1879 deduced empirically that the total radiant energy of all wavelengths emitted per unit area per second by a heated body is proportional to the fourth power of its absolute temperature. In 1884 Boltzmann derived the fourth power law by considering the black-body radiation as the working substance for the ideal Carnot Cycle. He showed that the law is strictly applicable to the radiation from a black body. The law is, therefore, generally called the Stefan-Boltzmann Law. The law can be obtained directly from Planck's radiation formula. The derivation is as follows:

The energy density of the total radiation of all wavelengths in a black body enclosure at temperature T is given by :

$$U = \int_0^\infty U_\lambda d\lambda = \int_0^\infty \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda kT} - 1} d\lambda$$

Let $$x = \frac{hc}{\lambda kT}$$

$\therefore$ $$\lambda = \frac{hc}{kTx}$$

and $$d\lambda = -\frac{hc}{kTx^2} dx$$

when $\lambda = 0$, $x = \infty$ and when $\lambda = \infty$, $x = 0$

Hence,

$$U = \int_\infty^0 \frac{8\pi hc}{1} \left(\frac{kTx}{hc}\right)^5 \frac{1}{e^x - 1} \left(-\frac{hc}{kTx^2} dx\right)$$

$$= -\int_\infty^0 \left(\frac{8\pi k^4 T^4}{h^3 c^3}\right) \frac{x^3}{e^x - 1} dx$$

$$= \frac{8\pi k^4 T^4}{h^3 c^3} \int_0^\infty \frac{x^3}{e^x - 1} dx$$

The value of the integral is $\frac{\pi^4}{15}$

$\therefore$ $$U = \frac{8\pi k^4 T^4}{h^3 c^3} \cdot \frac{\pi^4}{15} = \frac{4}{c}\left(\frac{2}{15} \frac{\pi^5 k^4}{h^3 c^2}\right) T^4 \quad ...(8)$$

It can be shown that for any black body the total radiation of all wavelengths emitted per unit area per second at a given temperature, *i.e.* the total emissive power E is related to U at the same temperature by the equation:

$$E = \frac{cU}{4}, \quad \text{or} \quad U = \frac{4E}{c} \quad ...(9)$$

Substituting for U in Eq. (8), we get

$$E = \left(\frac{2\pi^5 k^4}{15 h^3 c^2}\right) T^4 \quad ...(10)$$

or $$E = \sigma T^4 \qquad \text{... (11)}$$

where $$\sigma = \frac{2\pi^5 k^4}{15h^3c^2} \qquad \text{... (12)}$$

Eq. (11) is the Stefan-Boltzmann law of radiation. The constant σ is called Stefan's constant. The experimental value of this constant is

$$\sigma = 5.67 \times 10^{-8}\, \text{W/m}^2\text{K}^4.$$

When the value $\sigma = 5.79 \times 10^{-8}\, \text{W/m}^2\text{K}^4$ (this was the value known at that time), $k = 1.38 \times 10^{-23}$ J/K, $c = 3 \times 10^8$ m/s are substituted in Eq. (12), the value of h is found to be 6.57×10^{-34} Js. This was the first calculated value of h. The recent recommended value of h which is now widely used is :

$$6.626 \times 10^{-34}\, \text{Js}$$

this value of h is substituted in Eq. (5) for obtaining the values of U_λ, it is found that the theoretical distribution curves agree excellently with the experimental curves over the whole range of wavelengths.

The success of Planck's hypothesis in explaining the distribution of energy in the spectrum of black body was the beginning of quantum mechanics, We now describe some more important phenomena which are explained by this hypothesis.